JHEP HYPER VERSION

Received: *June 19, 2000*, Accepted: *October 25, 2000*
Revised: *October 17, 2000*

Hamiltonian systems with boundaries

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Abstract: Lately, to provide a solid ground for quantization of the open string theory with a constant B-field, it has been proposed to treat the boundary conditions as hamiltonian constraints. It seems that this proposal is quite general and should be applicable to a wide range of models defined on manifolds with boundaries. The goal of the present paper is to show how the boundary conditions can arise as constraints in a purely algebraic fashion within the hamiltonian approach without any reference to the lagrangian formulation of the theory. The construction of the boundary Dirac brackets is also given and some subtleties are pointed out. We consider four examples of field theories with boundaries: the topological sigma model, the open string theory with and without a constant B-field and electrodynamics with topological term. A curious result for electrodynamics on a manifold with boundaries is presented.

Keywords: Field Theories in Lower Dimensions, Chern-Simons Theories, Bosonic Strings.

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Contents

1. Introduction and motivation

Lately, there has been a renewed interest in field theories on manifolds with boundaries. In general one would expect a nontrivial relation between bulk and boundary dynamics. One such model is the open string theory with a constant B-field. It turns out that the bulk and boundary properties of this model are quite different [1]. Recently, in the attempt to provide a solid ground for the quantization of the model, it has been proposed to treat boundary conditions as hamiltonian constraints within the Dirac approach [2]–[6]. This idea seems quite powerful and could be applied to a wide range of models.

Let us recall that for systems with boundaries, traditionally the boundary conditions have to be imposed to properly define the functional derivatives in the theory. Specifically, for the hamiltonian treatment one needs a boundary conditions for the proper definition of the Poisson brackets (symplectic structure on the phase space). However, there is an alternative algebraic approach to the definition of symplectic structures. Using three basic properties of the Poisson bracket (antisymmetry, Leibniz rule and Jacobi identity) and the canonical brackets for momenta and coordinates one can calculate any bracket. Of course a function on the phase space should now be understood as a formal power expansion in momenta and coordinates. This approach to the Poisson bracket is in the spirit of quantum mechanics where the algebraic definitions are the basic ones.

Thus applying the algebraic definition of Poisson bracket one does not have to impose the boundary conditions in order to do concrete calculations. As a matter of fact one sees that to define momentum, hamiltonian and primary constraints formally in many models there is no need to use the boundary conditions. Thus the natural question arises: what is the status of boundary conditions in this framework. In this paper we try to answer this question. The main point which we are going to make is that the boundary conditions can arise in a purely algebraic fashion as hamiltonian constraints localized on the boundary. As a result all the Dirac machinery may be applied[1] to the boundary conditions vs. the hamiltonian boundary constraints. Indeed we need those boundary constraints to make the whole hamiltonian treatment consistent.

Let us make a few technical remarks. The basic idea is rather naive. Since one can formally define the momentum, hamiltonian, the primary constraints and do calculations with the Poisson brackets without using the boundary conditions we may proceed in a formally along Dirac's lines using the canonical Poisson brackets [7]. However now we are not allowed to throw away the total derivative terms (the boundary terms). These boundary terms produce the corresponding constraints on the boundary. The usual consistency conditions have to be required for these constraints. To handle the technical side of the idea it is useful to work with constraints Φ smeared with test functions $N(x)$

$$\Phi[N] = \int d^d x\, \Phi(x)\, N(x)\,. \tag{1.1}$$

This is often a convenient notation, especially when one wants to keep track of partial integrations in a calculation. For the case of boundary constraint one assumes that the smearing function has support on the boundary only. (Those assumptions do not effect the formal calculations in any way.) In all calculations we will avoid the functional questions and will concentrate attention on the algebraic aspect of the computations. We will see that there is no ambiguity as soon as a calculation is done using test functions. However to define the Dirac brackets one has to do calculations without the test functions (as far as the author knows) and this may lead to trouble in some cases. To make sense of those boundary Dirac's bracket in certain cases one should give a mathematically rigorous definition of the relevant objects. We do not do this and just present the formal answer with short comments. We clarify this point by considering concrete examples.

It is worthwhile to make some remarks concerning the status of boundary conditions also within the lagrangian formalism. The presence of a boundary can spoil some properties which hold in the situation without a boundary. For instance, two classically equivalent actions (in the sense that they reproduce the same equations of motion) can give rise to different boundary conditions. A nontrivial example of

[1]That is true up to certain technical problems which, we believe, can be resolved.

this situation is the relation between the Howe-Tucker and Nambu-Goto actions for the open branes. These actions give slightly different boundary conditions. To relate them is a subtle task and depends on the dimension of the background space-time.[2] This is another reason for looking at the hamiltonian treatment of the boundary conditions.

The main motivation behind the present work is to understand the status of the boundary conditions in a quantum theory especially an interacting one. The hamiltonian approach has certain advantages when it comes to quantizing a theory. (At least in principle it is clear what one should do.) At the end we will comment on the possible quantum applications of our results.

Let us briefly comment on the literature. In mathematical physics the hamiltonian systems with boundaries is an old subject (see [8] for a list of references). The main attention has been on possible modifications of the Poisson bracket by surface terms to fulfil general axiomatic properties in the presence of boundaries. A modifed Poisson bracket was defined in [8] and later generalized in [9]. Unlike [8, 9] the present discussion is not formal. Our attitude is conservative. We calculate the Poisson brackets in the standard way and keep track of the boundary terms which we interpret as hamiltonian constraints. Eventually these hamiltonian boundary constraints should lead to a Dirac bracket which gives the right symplectic structure for the model.

Since the general idea by itself is simple one and it is difficult to give any general theorems, examples are quite helpful. Thus throughout the paper we consider four examples: the topological sigma model with a boundary, the open string theory with and without a constant B-field and four dimensional U(1) gauge theory on a manifold with a boundary. There is a section for every example and at the end we summarize the results and discuss the problems. In the first section we consider the topological sigma model with boundaries. This example is rather simple and particular. It demonstrates that there is a difference between the functional and algebraic approaches to the Poisson bracket and that the former approach misses some interesting information about the boundary. In next two section we consider the open string theory. We show that the boundary constraints can arise in a purely algebraic fashion from the algebra of constraints. We hope that the discussion in the fourth section will clarify some points in earlier analysises of the problem [2]–[6]. We also point out that some problems may arise in the definition of modified symplectic structure on the boundary. The last section is devoted to euclidean electrodynamics with a topological term on a manifold with boundaries. In the spirit of open string theory with B-field we derive the modification of the symplectic structure on the boundary.

[2]For instance, the strings in two dimensional space-time: The Nambu-Goto action is linear and it gives rise only the Dirichlet boundary conditions. However the Howe-Tucker (Polyakov) action is quadratic and it might produce as well the Neumann boundary conditions.

2. Topological sigma model with boundaries

Let us consider a topological sigma model with boundaries defined on 2d-dimensional smooth manifold which admits a symplectic structure[3] ω

$$S = \frac{1}{2}\int_{\Sigma} d^2\xi\, \omega_{\mu\nu}(X)\partial_\alpha X^\mu \partial_\beta X^\nu \epsilon^{\alpha\beta}\,, \tag{2.1}$$

where Σ is a two-dimensional world-sheet with boundary. In the bulk this model is purely topological and has no local degrees of freedom [10]. For the present purposes we ignore the topological aspects of the model and thus assume that the background space-time manifold can be covered by one patch. It means that we can think of the sympelctic form as an exact two form $\omega = dA$ and the action (2.1) becomes

$$S = \int_{\partial\Sigma} d\tau A_\mu(X)\dot{X}^\mu\,, \tag{2.2}$$

which describes the boundary dynamics. These boundary dynamics are trivial ($\dot{X}^\mu = 0$) and there is just a modification of the Poisson brackets for X^μ since there are 2d second class constraints.

Now we want to try to extract the information about boundary dynamics from the hamiltonian treatment, starting from the action (2.1). A variation of the action (2.1) gives

$$\delta S = \int_{\partial\Sigma} d\xi^\alpha \omega_{\mu\nu}\delta X^\mu \partial_\alpha X^\nu + \frac{1}{2}\int_{\Sigma} d^2\xi\, (d\omega)_{\mu\nu\rho}\partial_\alpha X^\mu \partial_\beta X^\nu \epsilon^{\alpha\beta}\delta X^\rho\,, \tag{2.3}$$

where the last term vanishes by itself (see footnote) and the boundary term should vanish as well. Since $\omega_{\mu\nu}$ is nondegenerate, one should impose the Dirichlet boundary condition

$$\delta X^\mu|_{\partial\Sigma} = 0\,, \tag{2.4}$$

which simply means that the naive functional derivative with respect to X is not defined on the boundary. Thus the functional approach cannot be used to find the boundary dynamics. Instead we may use an algebraic approach to the problem. The action (2.1) produces $2d$ constraints

$$\Phi_\mu[N^\mu] = \int d\sigma N^\mu (P_\mu - \omega_{\mu\nu}(X)X'^\nu)\,, \tag{2.5}$$

which give the following Poisson bracket algebra

$$\{\Phi_\mu[N^\mu], \Phi_\nu[M^\nu]\} = \int d\sigma\, [(d\omega)_{\mu\nu\rho}N^\mu M^\nu X'^\rho] - N^\mu M^\nu \omega_{\mu\nu}|_0^\pi\,, \tag{2.6}$$

where N^μ and M^ν are test functions and $\sigma \in [0,\pi]$. In the present calculation (and as well as in the next two sections) we use the following formula

$$\int_0^\pi d\sigma \int_0^\pi d\sigma' f(\sigma)g(\sigma')\partial_\sigma(\delta(\sigma-\sigma')) = f(\sigma)g(\sigma)|_0^\pi - \int_0^\pi d\sigma f'(\sigma)g(\sigma)\,. \tag{2.7}$$

[3] $\omega = \omega_{\mu\nu}dX^\mu \wedge dX^\nu$ is a symplectic structure if $d\omega = 0$ and $\omega_{\mu\nu}$ is not degenerate.

which can be easily motivated. The constraints (2.5) are first class in the bulk and second class constraints on the boundary. To simplify the calculations we can do the following. Since we are working on one patch one can assume that the symplectic structure $\omega_{\mu\nu}$ is a constant matrix of a special form (due to the Darboux theorem in one patch there are always special coordinates where the symplectic form can be brought to canonical form). Thus because of (2.6) there is a suitable modification of the symplectic structure on the boundary

$$\{X^\mu, X^\nu\}|_{\partial\Sigma} = \omega^{\mu\nu}\,, \tag{2.8}$$

where $\omega^{\mu\nu}\omega_{\nu\rho} = \delta^\mu_\rho$. Furthermore,

$$\{P_\mu, P_\nu\}|_{\partial\Sigma} = -\frac{1}{4}\omega_{\mu\nu}\,, \qquad \{X^\mu, P_\nu\}|_{\partial\Sigma} = \frac{1}{2}\delta^\mu_\nu\,, \qquad \{X'^\mu, X^\nu\}|_{\partial\Sigma} = \frac{1}{2}\omega^{\mu\nu}\,. \tag{2.9}$$

It is easy to check that all these brackets have the desired properties (everything should have a trivial bracket with constraints on the boundary). Proceeding along standard lines one finds that $\delta X^\mu = \dot{X}^\mu$ equals N^μ in the bulk and zero on the boundary.

The present model is trivial, nevertheless it contains the essence of the general situation of hamiltonian models with boundaries. It shows that there is a difference between the functional and the algebraic approaches to the symplectic structure. In the algebraic approach the right boundary conditions arise by themselves in a consistent way. In the next sections we consider less trivial examples of this situation.

3. Open string theory without a B-field

Let us consider an open string theory in a flat space-time ($\eta_{\mu\nu} = (-1, 1, \ldots, 1)$) without antisymmetric background field. The model has the following action

$$S = -\frac{1}{2}\int d^2\sigma\, \sqrt{-h}\, h^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}\,, \tag{3.1}$$

where $h^{\alpha\beta}$ is an auxilary metric. The treatment of the theory is presented in string theory textbooks (for instance [11]). To the author's knowledge the canonical treatment of the open string has only been given in the lectures by Henneaux [12]. In this section we would like to have a new look at some well-known facts about open strings. The action (3.1) produces the following boundary condition

$$\left(\sqrt{-h}h^{10}\dot{X}^\mu + \sqrt{-h}h^{11}X'^\mu\right)\Big|_{0,\pi} = 0\,. \tag{3.2}$$

If one starts from the Nambu-Goto action instead then the general boundary condition is $\dot{X}^\mu \sim X'^\mu|_{0,\pi}$ which is equivalent to (3.2). The condition (3.2) can be rewritten as follows in phase space

$$\left(\eta_{\mu\nu}X'^\nu + \sqrt{-h}h^{01}P_\mu\right)\Big|_{0,\pi} = 0\,, \tag{3.3}$$

which states that $\eta_{\mu\nu}X'^\nu$ and P_μ are proportional to each other on the boundary.

Now let us turn to the hamiltonian analysis of the system. For the model (3.1) the constraints are well known

$$\mathcal{H}_1[N] = \int_0^\pi d\sigma P_\mu X'^\mu N\,, \qquad \mathcal{H}[M] = \int_0^\pi d\sigma (P_\mu \eta^{\mu\nu} P_\nu + X'^\mu \eta_{\mu\nu} X'^\nu) M\,, \tag{3.4}$$

and they hold at all points including the boundary. Since the system is generally covariant the naive hamiltonian vanishes identically. Both constraints (3.4) are first class and they correspond to reparametrizations of the two dimensional world sheet. The constraints obey the following Poisson bracket algebra

$$\begin{aligned}
\{\mathcal{H}_1[N], \mathcal{H}_1[M]\} &= \mathcal{H}_1[NM' - N'M] \\
\{\mathcal{H}_1[N], \mathcal{H}[M]\} &= \mathcal{H}[NM' - N'M] + NM\,(P_\mu \eta^{\mu\nu} P_\nu - X'^\mu \eta_{\mu\nu} X'^\nu)|_0^\pi \\
\{\mathcal{H}[N], \mathcal{H}[M]\} &= \mathcal{H}_1[4(NM' - N'M)]\,.
\end{aligned} \tag{3.5}$$

The bracket between $\mathcal{H}_1$ and $\mathcal{H}$ gives rise the boundary term which should be set to zero to make the hamiltonian treatment consistent. Since $\mathcal{H}_1$ and $\mathcal{H}$ hold everywhere we must require the following constraints on the boundary

$$P_\mu \eta^{\mu\nu} P_\nu|_{0,\pi} = 0\,, \qquad X'^\mu \eta_{\mu\nu} X'^\nu|_{0,\pi} = 0\,, P_\mu X'^\mu|_{0,\pi} = 0\,. \tag{3.6}$$

One might call them the boundary constraints. The next step should be to check whether the algebra of new constraints is closed or not. As we said before all calculations can be done in a formal way avoiding questions of regularization. For example let us introduce the following notation for the boundary constraints

$$\phi_1[N] = \int_0^\pi d\sigma\, N\, P_\mu \eta^{\mu\nu} P_\nu\,, \qquad \phi_2[M] = \int_0^\pi d\sigma\, M\, X'^\mu \eta_{\mu\nu} X'^\nu\,, \tag{3.7}$$

where N and M might be thought as test functions localized on the boundary (or around boundary if there is some regularization assumed). This kind of assumptions does not effect the formal calculations. For instance we calculate the following brackets

$$\{\phi_1[N], \phi_2[M]\} = 4\int_0^\pi d\sigma\, NM' P_\mu X'^\mu + 4\int_0^\pi d\sigma\, NM P_\mu X''^\mu - 4NMP_\mu X'^\mu|_0^\pi\,, \tag{3.8}$$

and see that secondary constraints arise. However the constraints (3.6) can be resolved since P and X' are null vectors on the boundary and they are orthogonal to each other there is a proportionality relation on the boundary

$$(\alpha P_\mu + \eta_{\mu\nu} X'^\nu)_{0,\pi} = 0\,, \tag{3.9}$$

where α is some proportionality constant which is subject to gauge condition (since it relates world-sheet density to the world-sheet vector). The conditions (3.9) give us the same information as one would get from the lagrangian formalism (3.3). Hence

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the whole system can be described as two first class constraints $\mathcal{H}_1$, $\mathcal{H}$ plus a set of second class boundary constraints (3.9). The constraints (3.9) are second class because of the non vanishing brackets

$$\{\Phi_\mu[N^\mu], \Phi_\nu[M^\nu]\} = \alpha \int_0^\pi d\sigma \, [N^\nu M'^\mu - N'^\mu M^\nu]\eta_{\mu\nu} \,, \tag{3.10}$$

where $\Phi_\mu[N^\mu]$ is (3.9) smeared with the test function N^μ. Proceeding formally for the second class constraints (3.9) we define the corresponding Dirac brackets

$$\{X^\mu(\sigma), X^\nu(\sigma')\} = \frac{\alpha}{2}\eta^{\mu\nu}\frac{1}{\partial_\sigma}\delta(\sigma - \sigma') \,, \tag{3.11}$$

as well as the brackets

$$\{X^\mu(\sigma), P_\nu(\sigma')\} = \frac{1}{2}\delta^\mu_\nu \delta(\sigma - \sigma') \,, \qquad \{X'^\mu(\sigma), X^\nu(\sigma')\} = \frac{\alpha}{2}\eta^{\mu\nu}\delta(\sigma - \sigma') \,. \tag{3.12}$$

We are interested in the restriction of these brackets to the boundary and it is not clear how to find this, especially for the non-local bracket (3.11). The point is that this question can not be answered unless our description of the model is supplemented with a certain amount of additional information. The extra information concerns the restrictions on the behaviour of the fields in order to make operator ∂_σ invertable (in general there is a constant zero mode for this operator). Therefore to make further progress one needs more insight into the model. It would be interesting to quantize the free open string theory in a nonconformal gauge (where $\alpha \neq 0$) and calculate the commutators (3.11) and (3.12) explicitly on the boundary. Resolving this kind of questions can lead to the proper understanding of the foundations of Witten's open string field theory [13] where the noncommutativity of the ends of strings plays a crucial role.

4. Open string theory with a constant B-field

Now let us turn to the open string theory with a constant B-field. The model has the following action

$$S = -\frac{1}{2}\int d^2\sigma \left(\sqrt{-h}h^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} - \epsilon^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu B_{\mu\nu}\right). \tag{4.1}$$

This system has attracted much attention recently because of the noncommutative properties of the end points of the string. A treatment of the model has been given in [1] (also see [14] for the quite full list of references). Let us just recall that in the lagrangian formalism one should impose the boundary conditions

$$\left(\sqrt{-h}h^{10}\eta_{\mu\nu}\dot{X}^\nu + \sqrt{-h}h^{11}\eta_{\mu\nu}X'^\nu + B_{\mu\nu}\dot{X}^\nu\right)\Big|_{0,\pi} = 0 \,, \tag{4.2}$$

which have the following form in phase space

$$(B_\mu{}^\nu P_\nu + G_{\mu\nu}X'^\nu + \sqrt{-h}h^{01}P_\mu)|_{0,\pi} = 0\,, \tag{4.3}$$

where $G_{\mu\nu} = \eta_{\mu\nu} - B_{\mu\sigma}B^\sigma{}_\nu$. For the sake of simplicity we assume that B is a non-degenerate matrix (for the degenerate case one can easily generalize all the following arguments).

Now we turn to the hamiltonian formalism. In the usual fashion the constraints are

$$\mathcal{H}_1[N] = \int_0^\pi d\sigma P_\mu X'^\mu N\,, \tag{4.4}$$

$$\mathcal{H}[M] = \int_0^\pi d\sigma (P_\mu \eta^{\mu\nu} P_\nu - 2P_\mu B^\mu{}_\nu X'^\nu + X'^\mu G_{\mu\nu} X'^\nu) M\,. \tag{4.5}$$

These are first class constraints and they hold everywhere including at the boundary points. Next we calculate the algebra keeping track of the boundary terms. The constraints obey the following Poisson bracket algebra

$$\{\mathcal{H}_1[N], \mathcal{H}_1[M]\} = \mathcal{H}_1[NM' - N'M], \tag{4.6}$$

$$\{\mathcal{H}_1[N], \mathcal{H}[M]\} = \mathcal{H}[NM' - N'M] + NM(P_\mu \eta^{\mu\nu} P_\nu - X'^\mu G_{\mu\nu} X'^\nu)|_0^\pi, \tag{4.7}$$

$$\{\mathcal{H}[N], \mathcal{H}[M]\} = \mathcal{H}_1[4(NM' - N'M)]\,. \tag{4.8}$$

To make the theory consistient one should set the boundary term to zero. Since $\mathcal{H}_1$ and $\mathcal{H}$ hold everywhere there is a boundary constraint

$$X'^\mu (B_\mu{}^\nu P_\nu + G_{\mu\nu} X'^\nu)|_{0,\pi} = 0\,, \tag{4.9}$$

which is the difference between $\mathcal{H}_1$ and the boundary term in (4.7). One cannot solve the system as simply as before. Therefore we proceed along Dirac's lines [7]. We look at possible secondary and tertiary constraints and then try to separate them into first and second class constraints. Sometimes, before separating them into different classes it is helpful to solve some of them.

We thus have to calculate brackets of all constraints including the boundary one and see if new constraints arise. We will perform the calculations in a formal way and introduce the following notation for the boundary constraint

$$\Phi[N] = \int_0^\pi d\sigma\, N\, X'^\mu (B_\mu{}^\nu P_\nu + G_{\mu\nu} X'^\nu)\,, \tag{4.10}$$

where N is a test function. As a result of the computations some new constraints will arise. Let us look at some of them to see the pattern. We have

$$\{\Phi[N], \Phi[M]\} = \int_0^\pi d\sigma\, [NM' - N'M] X'^\mu B_\mu{}^\rho (B_\rho{}^\nu P_\nu + G_{\rho\nu} X'^\nu)\,. \tag{4.11}$$

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Introducing the following notation for the new constraint

$$\Phi_1[N] = \int_0^\pi d\sigma\, N X'^\mu B_\mu{}^\rho (B_\rho{}^\nu P_\nu + G_{\rho\nu} X'^\nu)\,, \tag{4.12}$$

we get

$$\{\Phi_1[N], \Phi_1[M]\} = \int_0^\pi d\sigma\, [NM' - N'M] X'^\mu B_\mu{}^\delta B_\delta{}^\sigma B_\sigma{}^\rho (B_\rho{}^\nu P_\nu + G_{\rho\nu} X'^\nu)\,, \tag{4.13}$$

and

$$\{\Phi_1[N], \Phi[M]\} = \int_0^\pi d\sigma\, [NM' - N'M] X'^\mu B_\mu{}^\delta B_\delta{}^\rho (B_\rho{}^\nu P_\nu + G_{\rho\nu} X'^\nu)\,, \tag{4.14}$$

and so on. This suggests the following boundary conditions

$$X'^\mu M_\mu{}^\sigma (B_\sigma{}^\nu P_\nu + G_{\sigma\nu} X'^\nu)|_{0,\pi} = 0\,, \tag{4.15}$$

where M is some power of B. Since B is nondegenerate and antisymmetric all these conditions can be replaced by the following one

$$(B_\mu{}^\nu P_\nu + G_{\mu\nu} X'^\nu + \beta P_\mu)|_{0,\pi} = 0\,, \tag{4.16}$$

where β is the coefficient of proportionality which is subject to a gauge condition (like α in the previous section). We will see that (4.16) are second class constraints. Introducing the notation

$$\mathcal{K}_\mu[N^\mu] = \int_0^\pi d\sigma\, N^\mu (B_\mu{}^\nu P_\nu + G_{\mu\nu} X'^\nu + \beta P_\mu) \tag{4.17}$$

it is easy to check the brackets

$$\{\mathcal{K}_\mu[N^\mu], \mathcal{K}_\nu[M^\nu]\} = \int_0^\pi d\sigma\, [N^\mu M'^\nu - N'^\nu M^\mu](B_\mu{}^\rho G_{\rho\nu} + \beta G_{\mu\nu})\,, \tag{4.18}$$

where we have nondegenerate matrix on the right-hand side. Therefore we conclude that to make the whole hamiltonian treatment consistent one must impose the boundary conditions (4.3) which play the role of second class constraint on the boundary. Otherwise the algebra (4.8) would not be closed. Thus the bracket algebra has to be modified on the boundary. The Poisson bracket must be replaced by the Dirac bracket. For example on the boundary the coordinates have the following bracket

$$\{X^\mu, X^\nu\}_{\partial\Sigma} = -B^\mu{}_\sigma (G^{-1})^{\sigma\nu} + \beta(\text{nonlocal part})\,, \tag{4.19}$$

where the non local part has the same structure as in the previous section. For the case $\beta = 0$ (for instance, conformal gauge or static gauge) the brackets (4.19) are well defined. A discussion of the modified brackets is given in [2]–[6].

5. Electrodynamics with topological term

As a last example we consider theory with a nonvanishing hamiltonian. We will take a look at four dimensional euclidean electrodynamics with a topological term. The action is defined by

$$S = \frac{1}{2g^2}\int_{\mathcal{M}} F \wedge *F + \frac{i\theta}{4\pi^2}\int_{\mathcal{M}} F \wedge F\,, \tag{5.1}$$

where we use differential forms. Equivalently, in components,

$$S = \frac{1}{4g^2}\int_{\mathcal{M}} d^4x\, F_{\mu\nu}F^{\mu\nu} + \frac{i\theta}{16\pi^2}\int_{\mathcal{M}} d^4x\, \epsilon_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma}\,. \tag{5.2}$$

The theory is defined on a manifold $\mathcal{M}$ with non empty boundary $\partial\mathcal{M}$. Since we are interested in the hamiltonian treatment we assume that $\mathcal{M} = R \times \Sigma$ where Σ is a spatial manifold. For the sake of simplicity we further assume that Σ is closed set in R^3 and thus it carries a flat metric. This assumption is not essential and the whole logic can be generalized to the general curved case.

Before looking at the hamiltonian formalism we briefly consider the lagrangian formalism. To the author's knowledge this system has not been separately studied, except in [16]. The action (5.1) gives the following equations of motion

$$d * F = 0\,, \tag{5.3}$$

which should be supplemented by the boundary condition

$$\int_{\partial\mathcal{M}} \delta A \wedge \left(\frac{1}{g^2} * F + \frac{i\theta}{2\pi^2}F\right) = 0\,. \tag{5.4}$$

To proceed further let us write (5.4) in components

$$-\int dt \int_{\partial\Sigma} d^2s \left[n^a\left(\frac{1}{g^2}E_a + \frac{i\theta}{2\pi^2}B_a\right)\delta A^0 - n_b\epsilon^{abc}\delta A_c\left(\frac{1}{g^2}B_a + \frac{i\theta}{2\pi^2}E_a\right)\right], \tag{5.5}$$

where we introduce the standard notation $E_a \equiv F_{0a}$ and $B_a = \frac{1}{2}\epsilon_{abc}F^{bc}$ and n^a is a vector normal to $\partial\Sigma$. Now it is straightforward to read off the boundary conditions

$$n^a\left(\frac{1}{g^2}E_a + \frac{i\theta}{2\pi^2}B_a\right)\Big|_{\partial\mathcal{M}} = 0 \qquad \text{or} \qquad \delta A^0|_{\partial\mathcal{M}} = 0\,, \tag{5.6}$$

$$n_b\epsilon^{abc}\left(\frac{1}{g^2}B_a + \frac{i\theta}{2\pi^2}E_a\right)\Big|_{\partial\mathcal{M}} = 0 \qquad \text{or} \qquad \delta A_c|_{\partial\mathcal{M}} = 0\,. \tag{5.7}$$

Thus one of these two sets of conditions should be imposed to make the lagrangian treatment consistent.[4]

[4] Apart from this one can think about other physical requirements such as absence of energy-momentum flow through the boundary. In fact the conservation of energy-momentum requires the conditions on the left hand side of (5.6) and (5.7) [17].

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Let us rewrite the boundary conditions (5.6), (5.7) in phase space. The momentum is defined as follows

$$\pi_a = \frac{1}{g^2}E_a + \frac{i\theta}{2\pi^2}B_a, \tag{5.8}$$

and there is the usual constraint $\pi_0 = 0$ which we will discuss later on. Using (5.8) the boundary conditions (5.6), (5.7) become

$$n^a \pi_a|_{\partial\mathcal{M}} = 0 \qquad \text{or} \qquad \delta A^0|_{\partial\mathcal{M}} = 0\,, \tag{5.9}$$

$$n_b \epsilon^{abc}\left(\frac{i\theta}{2\pi^2}g^2\pi_a + \left(\frac{1}{g^2} - \left(\frac{i\theta}{2\pi^2}\right)^2 g^2\right)B_a\right)|_{\partial\mathcal{M}} = 0 \qquad \text{or} \qquad \delta A_c|_{\partial\mathcal{M}} = 0\,. \tag{5.10}$$

There are one normal condition (on the left handside (5.9)) and two tangential conditions (on the left handside (5.10)). We will keep these in mind. We hope to find them as boundary constraints required to make the whole hamiltonian treatment consistent. Let us assume that one can choose a coordinate system such that the normal vector has the form $\vec{n} = (1, 0, 0)$.

We now turn to the hamiltonian treatment. Using (5.1) and (5.8) one defines the hamiltonian

$$H = \int_\Sigma d^3x \left[\frac{g^2}{2}\pi_a\pi^a - \frac{i\theta}{2\pi^2}g^2\pi_a B^a - \frac{1}{2}\left(\frac{1}{g^2} - \left(\frac{i\theta}{2\pi^2}\right)^2 g^2\right)B_a B^a + (\partial_a A_0)\pi^a\right]. \tag{5.11}$$

Thus one defines the hamiltonian H and primary constraint π_0 without using the boundary conditions. Introducing the notation

$$\Pi[\Lambda] = \int_\Sigma d^3x\, \Lambda(x)\pi^0(x)\,, \tag{5.12}$$

one has the following bracket

$$\{\Pi[\Lambda], H\} = \mathcal{G}[\Lambda] - \int_{\partial\Sigma} d^2s \Lambda(n^a\pi_a)\,, \tag{5.13}$$

where $\mathcal{G}[\Lambda]$ is the Gauss law constraint

$$\mathcal{G}[\Lambda] = \int_\Sigma d^3x\, \Lambda(x)\partial_a\pi^a(x)\,. \tag{5.14}$$

Consistency then implies that the right hand of (5.13) must be equal to zero. Thus we are getting the standard Gauss law and as well the boundary constraint $n^a\pi_a = 0$ (if one assumes that A^0 is zero on the boundary there is no boundary term in (5.13)).

Following the standard prescription one should look at the time evolution of Gauss law (5.14) and the new boundary constraint

$$\Phi_1[\Lambda] = \int_\Sigma d^3x\, \Lambda n^a\pi_a\,, \tag{5.15}$$

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where Λ can be thought as test function with support on $\partial\Sigma$. The formal computation gives us the following result

$$\{\mathcal{G}[\Lambda], H\} = \int_{\partial\Sigma} d^2s\, [n_b \partial_c \Lambda \epsilon^{abc}] \left(\frac{i\theta}{2\pi^2} g^2 \pi_a + \left(\frac{1}{g^2} - \left(\frac{i\theta}{2\pi^2}\right)^2 g^2 \right) B_a \right), \qquad (5.16)$$

$$\{\Phi_1[\Lambda], H\} = \int_{\Sigma} d^3x\, [\partial_c(\Lambda n_b) \epsilon^{abc}] \left(\frac{i\theta}{2\pi^2} g^2 \pi_a + \left(\frac{1}{g^2} - \left(\frac{i\theta}{2\pi^2}\right)^2 g^2 \right) B_a \right), \qquad (5.17)$$

where we have used the higher dimensional analog of equation (2.7). The bracket (5.16) is localized on the boundary and it gives us the tangential boundary constraints which exactly coincide with the boundary conditions (5.10). The same boundary constraint is given by the bracket (5.17) and it is localized on the boundary since Λ has support on the boundary only. Therefore we introduce the new boundary constraint

$$\Phi_a[N^a] = \int d^3x\, N_a \left[\frac{i\theta}{2\pi^2} g^2 \pi^a + \left(\frac{1}{g^2} - \left(\frac{i\theta}{2\pi^2}\right)^2 g^2 \right) B^a \right], \qquad (5.18)$$

where it is assumed that $N_a = (0, N_2, N_3)$. To decide on the status of boundary constraints Φ_1, Φ_2 and Φ_3 one should calculate the following brackets

$$\{\Phi_1[\Lambda], \Phi_b[N^b]\} = \left(\frac{1}{g^2} - \left(\frac{i\theta}{2\pi^2}\right)^2 g^2 \right) \int_{\Sigma} d^3x\, \Lambda n^a \partial^b N^c \epsilon_{abc}\,, \qquad (5.19)$$

$$\{\Phi_a[N^a], \Phi_b[M^b]\} = \frac{i\theta}{2\pi^2} g^2 \left(\frac{1}{g^2} - \left(\frac{i\theta}{2\pi^2}\right)^2 g^2 \right) \int_{\Sigma} d^3x\, \partial^a (N^b M^c) \epsilon_{abc}\,. \qquad (5.20)$$

One notices that the brackets (5.20) of Φ_a ($a = 2, 3$) are non-zero because of the boundary term on the right hand side of (5.20). In the bulk such brackets are zero since the constraints Φ_a are generalization of the chiral condition for two forms [15]. Therefore the brackets of all boundary constraints give a field independent anti-symmetric matrix with rank 2. It turns out that there is one first class boundary constraint $n^a \pi_a$ which makes us able to gauge away the normal component of the connection on the boundary. The boundary constraints Φ_2 and Φ_3 are second class constraints which lead to the the following Dirac bracket on the boundary

$$\{A_2(x), A_3(y)\}|_{\partial\Sigma} = \frac{i\theta}{2\pi^2} g^2 \left(\frac{1}{g^2} + \frac{\theta^2}{4\pi^4} g^2 \right)^{-1} \delta^{(2)}(x-y)\,. \qquad (5.21)$$

In analogy with the models considered in the previous section we see that at the boundary the Poisson brackets should be replaced by the corresponding Dirac bracket. In general the boundary Dirac bracket will depend on the geometry of the boundary. We will discuss this elsewhere. The physical interpretation of (5.21) is unclear. It seems that one will have problems with localizing photons on the boundary. Certainly this subject deserves an independent study [17] and we do no analyse the boundary theory further here.

6. Discussion and problems

In this paper we made an attempt to understand the status of the boundary conditions within the hamiltonian formalism motivated by the quantum theory. We have shown that boundary conditions can arise in a purely algebraic fashion as hamiltonian boundary constraints. Their existence is necessary to make the whole hamiltonian treatment consistent. Our arguments were based on four examples: the topological sigma model, the open string theory with and without a B-field and electrodynamics with a topological term. For some systems it is important to motivate that the boundary conditions can be treated as hamiltonian constraints. This type of systems has non trivial boundary conditions which mix momenta and coordinates. Such boundary conditions change the canonical brackets on the boundary drastically and therefore they are very important for the quantization of the system as whole. However as we saw in some instances (e.g., (3.11)) problems can arise with the definition of the Dirac bracket on the boundary. To resolve those problems one needs more insight into the models. In other cases there is no ambiguity in defining the modified symplectic structure (e.g., (2.8), local part of (4.19) and (5.21)).

As can be seen from the last example the boundary conditions give rise not only to second class constraints but also to first class constraint. It is unclear how this kind of boundary constraint should be applied in the quantum theory. We hope to return to this question and do explicit calculations for this model in the presence of a simple boundary.

From a technical point of view the present approach is based on rules for dealing with the following expression

$$\int_\Sigma d^d x \int_\Sigma d^d x' f(x) g(x') D_x D_{x'} \delta(x - x')\,, \tag{6.1}$$

where D_x ($D_{x'}$) is some differential operator. As was pointed out in [8] the expression (6.1) is not in general well defined on a closed domain. In the present paper we considered simple models with a few derivatives and therefore it was no problem to define (6.1) in a reasonable way. However in general one should address this question more carefully.

It would be interesting to study an interacting theory like Yang-Mills theory and gravity systems in this context. We hope to treat these questions elsewhere.

Acknowledgments

It is pleasure to thank Ingemar Bengtsson, who has promoted my interest in the subject and who has helped a lot during the preparation of this work. I am grateful to Ingermar Bengtsson and Ulf Lindström for reading and commenting on the manuscript. I thank M.M. Sheikh-Jabbari and V.O. Soloviev for bringing the relevant references to my attention.

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